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GROUPS OF HOMOTOPY EQUIVALENCES

BY

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The set of homotopy classes of homotopy self-equivalences of a topological space forms a group with a multiplication induced from composition. This group is important in algebraic topology, both for aesthetic reasons and because of its connection with the problem of finding a complete set of homotopy invariants. (In general, a Postnikov system overdetermines the homotopy type of a space.) Nevertheless, relatively little work has been done towards determining these groups. They are known, of course, for spheres and Eilenberg-MacLane spaces, and they have been studied in certain special cases by Barcus and Barratt (2).

The primary purpose of this paper is to obtain some information on these groups in general. The method is inductive, and it is connected with the Postnikov procedure of decomposing a space according to homotopy groups. I actually consider two different groups of homotopy equivalences, the first being the ordinary one and the second being a quotient of it by those equivalences which induce the identity isomorphism on homotopy groups. In the absence of any structure theory for non-Abelian groups (which may possibly be infinitely generated), it seems impossible to determine these groups exactly. However, I can determine them, up to a series of extensions. The organization of the paper is as follows: section 1 contains some preliminaries and the basic definitions. In section 2, I establish the basic exact sequences.

The section 3 contains some elementary applications, as well as a determination of the groups in some special cases (such as the terms in a Postnikov decomposition of a sphere).

1. Preliminaries

Throughout this paper, all spaces will have a base point, which will be preserved by maps (and homotopies) unless otherwise mentioned. For sections 2 and 3, we shall have to assume that the spaces have the homotopy type of a 1-connected complex. We denote by $\mathcal{G}(X)$ the group of homotopy class of homotopy equivalences from X to X.

I do not know what functorial properties $\mathcal{G}(X)$ has in general. However, we have the following elementary fact.

Prop. 1.1 $\mathcal{G}$ is a contravariant functor on the category of spaces which have the same homotopy type as a fixed space X and maps which are homotopy equivalences. That is, if $f : X \longrightarrow Y$ is a homotopy equivalence, there is a homomorphism (2)
$\mathcal{G}(f) : \mathcal{G}(Y) \longrightarrow \mathcal{G}(X)$ such that
1.) $\mathcal{G}(\mathrm{Id.}) = \mathrm{Id.}$
2.) If $g : Y \longrightarrow Z$ is a homotopy equivalence,
$\mathcal{G}(g \cdot f) = \mathcal{G}(f) \cdot \mathcal{G}(g)$.

Proof: If $\{\alpha\} \in \mathcal{G}(Y)$, then set $\mathcal{G}(f)\{\alpha\} = \{f^{-1} \cdot \alpha \cdot f\}$.

Def. 1.1 Denote by $\mathcal{G}_I(X)$ the subset of $\mathcal{G}(X)$ consisting of those homotopy classes whose elements induce the identity isomorphism on the homotopy groups of X.

Prop. 1.2 $\mathcal{G}_I(X)$ is a normal subgroup of $\mathcal{G}(X)$.

Proof: $\mathcal{G}_I(X)$ is clearly a subgroup. If $\{\alpha\} \in \mathcal{G}_I(X)$ and $\{\gamma\} \in \mathcal{G}(X)$, it is clear that $\gamma \cdot \alpha \cdot \gamma^{-1}$ induces the identity isomorphism on homotopy groups, so that $\{\gamma\}\{\alpha\}\{\gamma^{-1}\} \in \mathcal{G}_I(X)$.

Def. 1.2 Denote $\mathcal{G}_N(X) = \mathcal{G}(X)/\mathcal{G}_I(X)$.

Remark: It is easy to construct examples for which

$$1 \neq \mathcal{G}_I(X) \neq \mathcal{G}(X) .$$

Now, suppose X is a fibre space over B (by which we mean that there is a map $p : X \longrightarrow B$ which satisfies the covering homotopy property for maps of any space into B). We consider homotopy equivalences of X which cover the identity map of B. That means we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

If α and α' are two such homotopy equivalences, then $\alpha' \circ \alpha$ again covers the identity. A homotopy inverse α^{-1} for a map α covering B is a map α^{-1} which covers B so that $\alpha^{-1} \cdot \alpha$ and $\alpha \cdot \alpha^{-1}$ are fibre homotopic to the identity.

Prop. 1.3 Fibre homotopy classes of homotopy equivalences α which cover the identity on B form a group, under composition.

We denote this group by $\mathcal{G}_F(X)$. There is a natural homomorphism

$$\rho_F : \mathcal{G}_F(X) \longrightarrow \mathcal{G}(X) .$$

Proof: Consider two maps α and α' as above, and denote their fibre homotopy classes by $\{\alpha\}$ and $\{\alpha'\}$. Clearly, $\alpha \cdot \alpha'$ is a homotopy equivalence which covers the identity. One easily checks that $\{\alpha' \cdot \alpha\}$ is independent of the choices of representative in $\{\alpha\}$ and $\{\alpha'\}$, so that these classes compose. Inverses exist by definition.

The map ρ_F associates with every fibre homotopy equivalence class the corresponding homotopy class.

Remark: In Prop. 1.3, I have assumed the existence of inverses for fibre homotopy equivalences, such that the two compositions are fibre homotopic to the identity. In the case which we shall consider in section 3, the existence of these inverses will follow from a theorem of A. Dold (5).

Next, we define Postnikov systems and discuss their basic properties.

Def. 1.3 Let X be a space (which is assumed to be connected). A Postnikov system for X consists of a family of spaces X_n, $n \geq 0$, maps $\pi_n : X_n \longrightarrow X_{n-1}$ and $p_n : X \longrightarrow X_n$, such that

1. If X is k-connected, X_i = point for $i \leq k$.
2. $\pi_n : X_n \longrightarrow X_{n-1}$ is a principal fibre map, with fibre $K(\pi_n(X), n)$.
3. p_n is an n-equivalence.
4. $\pi_n \cdot p_n \simeq p_{n-1}$.

We denote the system by $\{X_n, p_n, \pi_n\}$.

Then the following theorem is known (see (6) and (7)).

Theorem 1.1

Let X and X' be spaces of the homotopy type of 1-connected complexes. Let $f : X \longrightarrow X'$. Then there are maps $f_n : X_n \longrightarrow X'_n$, for any Postnikov systems $\{X_n, \pi_n, p_n\}$ and $\{X'_n, \pi'_n, p'_n\}$ for X and X' (resp.), such that

$$\pi'_n \cdot f_n = f_{n-1} \cdot \pi_n$$

and

$$p'_n \cdot f \simeq f_n \cdot p_n .$$

If the k-invariants (images of the fundamental classes under transgression) are denoted k^n and k'^n , we have

$$f^c_{n-1} \cdot k_n = f_{n-2}\, k'^n$$

where f^c_{n-1} is the coefficient homomorphism induced by the map $f_\# : \pi_{n-1}(X) \longrightarrow \pi_{n-1}(X')$.

If $f, g : X \longrightarrow X'$ and $f \simeq g$, then $f_n \simeq g_n$ for all $n \geq 0$. (In fact, any maps which satisfy the same conditions as f_n and g_n are homotopic.)

We then have

Cor. 1.1 If f is a homotopy equivalence, then each f_n is also a homotopy equivalence.

Prop. 1.4 Let X be as in Theorem 1.1. Then the correspondence $f \longrightarrow f_n$ defines homomorphisms

$$\begin{array}{ll} \rho : \mathcal{G}(X) \longrightarrow \mathcal{G}(X_n) & ; \ n \geq 0 \\ \rho_N : \mathcal{G}_N(X) \longrightarrow \mathcal{G}_N(X_n) & \end{array}$$

Proof: By Cor. 1.1, f_n is a homotopy equivalence. Since

f_n is determined up to homotopy by the homotopy class of f, the map ρ is well-defined. If $f \in \mathcal{G}_I(X)$, it is clear that $f_n \in \mathcal{G}_I(X_n)$. Hence, ρ_N is well-defined. We must show that ρ is a homomorphism. Let $f_1, f_2 : X \longrightarrow X$ be homotopy equivalences. Then, we have a homotopy commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, "p_n"] & X \arrow[r, "f_1"] \arrow[d, "p_n"] & X \arrow[d, "p_n"] \\
X_n \arrow[r, "(f_1)_n"] & X_n \arrow[r, "(f_2)_n"] & X_n
\end{tikzcd}$$

and a homotopy commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f_2 \cdot f_1"] \arrow[d] & X \arrow[d] \\
X_n \arrow[r, "(f_2 \cdot f_1)_n"] & X_n
\end{tikzcd}$$

Thus, as in Theorem 1.1, we must have $(f_2 \cdot f_1)_n \simeq (f_2)_n \cdot (f_1)_n$. (That is to say $(f_2 \cdot f_1)_n$ and $(f_2)_n \cdot (f_1)_n$ are both induced maps on X_n for $f_2 \cdot f_1$, and hence they are homotopic). (See (7)).

It follows that ρ_N is a homomorphism.

2. The Exact Sequences:

The purpose of this section is to show how one can determine the groups defined above, up to a sequence of extensions. The procedure is practical only when there are a finite number of extensions and only when certain information about the spaces in question is known. I will illustrate such cases in section 3. For the present section, recall that

$\mathcal{G}(K(\pi,m)) = \mathrm{Aut}(\pi)$, $\mathcal{G}_I(K(\pi,m)) = 1$, and $\mathcal{G}_N(K(\pi,m)) = \mathrm{Aut}(\pi)$. Thus, we know these groups for the first non-trivial term of a Postnikov system $\{X_n, p_n, \pi_n\}$ for a space X (which is assumed to have the homotopy type of a 1-connected countable complex). We shall now construct exact sequences which determine $\mathcal{G}(X_n)$ (or $\mathcal{G}_N(X_n)$) in terms of $\mathcal{G}(X_{n-1})$ and other information about the Postnikov system, at least up to extension. Thus, we shall determine the groups $\mathcal{G}(X_n)$ inductively, modulo some extensions.

In section 1, we defined a homomorphism $\rho : \mathcal{G}(X) \longrightarrow \mathcal{G}(X_n)$. Since for any fixed $i > 0$, the terms X_j, $j < i$ form a Postnikov system for X_i, we have homomorphisms $\rho : \mathcal{G}(X_i) \longrightarrow \mathcal{G}(X_j)$. We shall be particularly interested in

$$\rho^n : \mathcal{G}(X_n) \longrightarrow \mathcal{G}(X_{n-1}) ; \qquad n > 0 .$$

(We define ρ^n_N in a similar way.) We plan to analyze the kernel and image of this homomorphism.

We suppose that X is $(m-1)$-connected, and that $n > m$.

__Lemma 2.1__ Consider $\rho^n : \mathcal{G}(X_n) \longrightarrow \mathcal{G}(X_{n-1})$. We have

$$\mathrm{Im}(\rho^n) = \{\{f_{n-1}\} \in \mathcal{G}(X_{n-1}) \mid \exists\, f^c \in \mathrm{Aut}(\pi_n(X)) \ni f^c k^{n+1} = f^*_{n-1}\, k^{n+1}\}$$

where $k^{n+1} \in H^{n+1}(X_{n-1} ; \pi_n(X))$ is the k-invariant and f^c is to be interpreted as a coefficient homomorphism.

We denote $\mathrm{Im}(\rho^n) = F(X_{n-1})$.

__Proof:__ Consider the diagram

$$\begin{array}{ccccc}
K(\pi_n(X), n) & \xrightarrow{i} & X_n & & X_n & \xleftarrow{i} & K(\pi_n(X), n) \\
 & & \big\downarrow \pi_n & & \big\downarrow \pi_n & & \\
 & & X_{n-1} & \xrightarrow{f_{n-1}} & X_{n-1} & &
\end{array}$$

where i is the injection of the fibre. If there is a map $f_n : X_n \longrightarrow X_n$ which makes this diagram commutative, then as is shown in (6),

$$(f_n)^c_{\#} \; k^{n+1} = f^*_{n-1} \; k^{n+1} .$$

Hence, if $\{f_{n-1}\}$ is an image under ρ^n, it belongs to the right-hand side of the equation which we are proving. On the other hand, suppose $\{f_{n-1}\}$ belongs to the right-hand side. Then there is $f^c \varepsilon \operatorname{Aut}(\pi_n(X))$ so that $f^c \, k^{n+1} = f^*_{n-1} \, k^{n+1}$. Taking $f^c : K(\pi_n(X), n) \longrightarrow K(\pi_n(X), n)$ to be a fibre map, we may construct a commutative diagram

$$\begin{tikzcd} X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\bar{k}^{n+1}"] & X_{n-1} \arrow[d, "k^{n+1}"] \\ K(\pi_n(X),n) \arrow[r, "f^c"] & K(\pi_n(X),n) \end{tikzcd}$$

where $\bar{k}^{n+1}$ is homotopically equivalent to k^{n+1}. Regarding k^{n+1} and $\bar{k}^{n+1}$ as maps into classifying spaces, we get a commutative diagram

$$\begin{tikzcd} \bar{X}_n \arrow[r, "\bar{f}_n"] \arrow[d, "\bar{p}_n"] & X_n \arrow[d, "p_n"] \\ X_{n-1} \arrow[r, "f_{n-1}"] & X_{n-1} \end{tikzcd}$$

Varying k^{n+1} by homotopy changes, X_n by bundle equivalence. Hence, there is a homeomorphism h such that

$$\begin{tikzcd} X_n \arrow[rr, "h"] \arrow[dr, "p_n"'] & & \bar{X}_n \arrow[dl, "\bar{p}_n"] \\ & X_{n-1} & \end{tikzcd}$$

commutes, so that we may define $f_n = h \cdot \bar{f}_n$. Then f_n makes our original diagram commutative, so that $\{f_{n-1}\}$ is in the image of ρ^n . (f_n is a homotopy equivalence because it follows from the five-lemma that f_n induces isomorphisms on homotopy groups, and we may then apply Whitehead's theorem.)

The determination of $\ker(\rho^n)$ requires more work. Recall that (by Prop. 1.3) for a given Postnikov fibration $\pi_n : X_n \longrightarrow X_{n-1}$, we have a group of fibre homotopy classes of fibre homotopy equivalences. We note the following facts: Each $\pi_n : X_n \longrightarrow X_{n-1}$ may be taken as a principal fibre bundle with group $K(\pi_n(X),n)$, and paracompact base (the base may be built from path spaces). Each $\pi_n : X_n \longrightarrow X_{n-1}$ is locally trivial. We then have the following

Prop. 2.1 (A.Dold). If $f : X_n \longrightarrow X_n$ is a homotopy equivalence such that $\pi_n \cdot f = \pi_n$, then f is a fibre homotopy equivalence.

Proof: See Theorem 6.1 of (5).

As in Prop. 1.3, we conclude the following

Prop. 2.2 The set of fibre homotopy classes of homotopy equivalences $f : X_n \longrightarrow X_n$, such that $\pi_n \cdot f = \pi_n$, forms a group, which we denote $\mathcal{G}_F(X_n)$

Lemma 2.2 There is a homomorphism

$$\rho_F : \mathcal{G}_F(X_n) \longrightarrow \mathcal{G}(X_n)$$

whose image is $\ker(\rho)$.

Proof. If $\{f\} \in \mathcal{G}_F(X_n)$, f is a homotopy equivalence, whose homotopy class we denote by $\rho_F\{f\}$. Both $\rho_F\{f\} \cdot \rho_F\{g\}$

and $\rho_F\{f \cdot g\}$ are represented by $f \cdot g$, so that ρ_F is a homomorphism.

If $\{f\} \varepsilon \mathrm{Im}(\rho_F)$, $\{f\}$ contains a map $f \ni \pi_n f = \pi_n$. Thus, $\{f\} \varepsilon \ker(\rho)$. If $\{f\} \varepsilon \ker(\rho)$, $\pi_n f \simeq \pi_n$, and by the covering homotopy property, we may choose $\bar{f} \simeq f$ so that $\pi_n \cdot \bar{f} = \pi_n$. Thus, $\{f\} \varepsilon \mathrm{Im}(\rho_F)$.

__Lemma 2.3__ $\mathcal{G}_F(X_n)$ is isomorphic, as a set, to a subset of $H^n(X_n; \pi_n(X))$. The multiplication in $\mathcal{G}_F(X_n)$ is given explicitly by the action of the fibre $K(\pi_n(X),n)$ on X_n (see proof). The subset of $H^n(X_n; \pi_n(X))$ is determined by the inclusion of the fibre $i : K(\pi_n(X),n) \longrightarrow X_n$.

__Proof:__ Let $h : X_n \longrightarrow X_n$ be a fibre homotopy equivalence. Then, if $x \varepsilon X_n$, x and $h(x)$ belong to the same fibre. Hence, there is a unique $a \varepsilon K(\pi_n(X),n)$ such that $a \cdot x = h(x)$. The correspondence $x \longrightarrow a$ defines a map

$$h^* : X_n \longrightarrow K(\ _n(X),\ n)$$

so that $h^* * i(\chi) = h^*(i(\chi)) \cdot i(\chi)$ is a homotopy equivalence on $K(\pi_n(X),n)$. Conversely, if

$$g^* : X_n \longrightarrow K(\pi_n(X),n)$$

is such that $g^* * i$ is a homotopy equivalence, one easily checks that g is a homotopy equivalence, where $g(\chi) = g^*(\chi) \cdot \chi$.

It is easy to see that, under these two correspondences, homotopy goes over to fibre homotopy and visa versa. The correspondences are clearly inverses of one another, so that $\mathcal{G}_F(X)$ is in 1-1 correspondence with the subset of

$h^* \varepsilon H^n(X_n; \pi_n(X))$ such that $h^* * i$ is a homotopy equivalence.

Consider 2 fibre homotopy equivalences f and g. We may write $g \cdot f$ as

$$X_n \xrightarrow{f^* \times 1} K(\pi_n(X), n) \times X_n \xrightarrow{\bar{\mu}} X_n \xrightarrow{g^* \times 1} K(\pi_n(X), n) \times X_n \xrightarrow{\bar{\mu}} X_n$$

where $\bar{\mu}$ denotes the action of the fibre on X_n. This shows how the multiplication in $\mathcal{G}_F(X_n)$ viewed as a subset of $H^n(X_n; \pi_n(X))$, is determined by the action of the fibre.

<u>Lemma 2.4</u> $I(X_n) = \ker(\rho_F)$ is the group of those fibre homotopy equivalences which are globally homotopic to the identity. It is described by the action of $K(\pi_n(X), n)$ on X_n (see proof).

<u>Proof:</u> Let $\alpha \varepsilon \mathcal{I}(X_n)$. Then $\rho_F(\alpha) = \{Id.\}$, i.e. any $f \varepsilon \alpha$ is globally homotopic to the identity. Conversely, if $f \varepsilon \alpha$ is globally homotopic to the identity, $\alpha \varepsilon \ker(\rho_F)$.

As above, we may write f as the following composition

$$X_n \xrightarrow{f^* \times 1} K(\pi_n(X), n) \times X_n \xrightarrow{\bar{\mu}} X_n$$

Suppose we denote by $\sigma : H^*(X_n; \pi_n(X_n)) \longrightarrow \pi[X_n, X_n]$ the operation described by the preceeding formula. Then, we can say that $\{f\} \varepsilon \mathcal{I}(X_n)$, if and only if $\sigma(f^*) = \{Id.\}$.

Summarizing what we have proved, we have

<u>Theorem 2.1 :</u> Let the space X have a Postnikov system $\{X_n, \pi_n, p_n\}$. Then, for any $n \geq 3$, we have an exact sequence.

$$1 \longrightarrow \ker(\rho) \longrightarrow \mathcal{G}(X_n) \xrightarrow{\rho^n} \mathcal{F}(X_{n-1}) \longrightarrow 1$$

and an exact sequence

$$1 \longrightarrow \mathcal{I}(X_n) \longrightarrow \mathcal{G}_F(X_n) \xrightarrow{\rho_F} \ker(\rho) \longrightarrow 1$$

$\mathcal{G}_F(X_n)$ is in 1-1 correspondence with a subset of $H^n(X_n;\pi_n(X))$. The multiplication in $\mathcal{G}_F(X_n)$ is explicitly determined by the map $\bar{\mu} : K(\pi_n(X),n) \times X_n \longrightarrow X_n$. The subgroup $I(X_n)$ is also determined explicitly by $\bar{\mu}$.

Next, we consider the groups $\mathcal{G}_N(X_n)$. To simplify notation, let us agree to write a subscript I to indicate the subset of the group under consideration, consisting of elements which induce the identity isomorphism on homotopy groups. (4)

<u>Lemma 2.5</u> $\ker(\rho)_I$, $\mathcal{F}(X_{n-1})_I$, $\mathcal{I}(X_n)_I$ and $\mathcal{G}_F(X_n)_I$ are normal subgroups of the appropriate groups. $\mathcal{I}(X_n)_I = \mathcal{I}(X_n)$.

<u>Proof:</u> They are all clearly subgroups. If $\alpha \in \ker(\rho)_I$ and $\beta \in \ker(\rho)$, then $\beta^{-1}\cdot\alpha\cdot\beta \in \ker(\rho)$, while on homotopy groups $(\beta^{-1}\cdot\alpha\cdot\beta)_\# = (\beta^{-1})_\#\cdot(\alpha)_\#\cdot(\beta)_\# = (\beta^{-1})_\#\cdot(\beta)_\# = 1$. The rest is similar.

<u>Lemma. 2.6</u> The following diagrams are commutative. (4)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker(\rho)_I & \longrightarrow & \mathcal{G}(X_n)_I & \longrightarrow & \mathcal{F}(X_{n-1})_I & \longrightarrow & 1 \\
\| & & \cap & & \cap & & \cap & & \| \\
1 & \longrightarrow & \ker(\rho) & \longrightarrow & \mathcal{G}(X_n) & \longrightarrow & \mathcal{F}(X_{n-1}) & \longrightarrow & 1
\end{array}$$

$$\begin{tikzcd}
 & & \mathcal{G}_F(X_n)_I \arrow[r] \arrow[d, phantom, "\cap"] & \ker(\rho)_I \arrow[d, phantom, "\cap"] \arrow[dr] & \\
1 \arrow[r] & I(X_n) \arrow[ur] \arrow[r] & \mathcal{G}_F(X_n) \arrow[r] & \ker(\rho) \arrow[r] & 1
\end{tikzcd}$$

<u>Proof:</u> The maps in the top rows are simply restrictions of the corresponding maps in the bottom rows.

Putting these facts together, we get from Lemmas 2.1, 2.5 and 2.6 and Theorem 2.1, the following

<u>Theorem 2.2</u> Keep the notation of Theorem 2.1. For each $n \geq 3$, we have an exact sequence

$$1 \longrightarrow \ker(\rho)_I \longrightarrow \mathcal{G}(X_n)_I \longrightarrow \mathcal{F}(X_{n-1})_I \longrightarrow 1$$

Now, it is well known that the nine-lemma is valid on the category of groups, i.e.

<u>Lemma 2.7</u> Let $A' \subset A$, $B' \subset B$, $C' \subset C$ be normal subgroups. Suppose we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & A/A' & \longrightarrow & B/B' & \longrightarrow & C/C' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

in which the columns are the usual exact sequences. If the first two rows are exact, then so is the third.

<u>Theorem 2.3</u> Keep the notation of Theorem 2.1. Then, for each $n \geq 3$, we have an exact sequence

$$1 \longrightarrow \ker(\rho)/\ker(\rho)_I \longrightarrow \mathcal{G}_N(X_n) \longrightarrow \mathcal{F}(X_{n-1})/\mathcal{F}(X_{n-1})_I \longrightarrow 1$$

<u>Proof:</u> Apply lemma 2.7 to lemma 2.6 and Theorem 2.1 and 2.2.

<u>3. Applications</u>[3]

We begin the applications with a sequence of finiteness

theorems. While they are straightforward applications of the results of section 2, they do not seem to be mentioned in the literature. We start with an elementary lemma.

<u>Lemma 3.1</u> If π is a finite (resp. fintely-generated) Abelian group, and X is an Eilenberg-MacLane space $K(\pi, n)$, then $\mathcal{G}(X)$ is a finite (resp. finitely-generated) group.

<u>Proof:</u> We have $\mathcal{G}(X) = \mathrm{Aut}(\pi)$. If π is finite, $\mathrm{Aut}(\pi)$ is finite. If π is finitely-generated, we may write

$$= Z + \ldots + Z + T$$

where T is a finite group. It is clearly sufficient to show that $\mathrm{Aut}(Z + \ldots + Z)$ is finitely-generated. But this amounts to asking if the group of integer-valued matrices with determinant of absolute value 1 is finitely-generated. This last fact is known.[(5)] (See, for example, (3)).

<u>Theorem 3.1</u> If X has finitely-many non-zero homotopy groups, each of which is finite, then $\mathcal{G}(X)$ is finite.

<u>Proof:</u> Let $\{X_n, p_n, \pi_n\}$ be a Postnikov system for X . If $X_m = K(\pi_m(X)\ m)$ is the first non-trivial term, then by lemma 3.1, $\mathcal{G}(X_m)$ is finite. We assume, for induction, that $\mathcal{G}(X_{n-1})$ is finite. Then, $\mathcal{F}(X_{n-1})$ is finite. It is clear that $H^n(X_n;\pi_n(X))$ is finite, so that by Theorem 2.1, $\ker(\rho)$ and hence $\mathcal{G}(X_n)$ is finite.

<u>Remark:</u> I do not know whether Theorem 3.1 is true if one replaces the word finite by finitely-generated. This theorem is clearly the crudest theorem one could state in this case. Actually,

using Theorem 2.1, one can derive information about the order of elements in $\mathcal{G}(X)$. Details are left to the reader.

In the same spirit, we have

<u>Theorem 3.2</u> Let X be an Abelian topological group. We suppose that

1.) The first non-trivial homotopy group is finitely-generated.

2.) There are finitely-many remaining homotopy groups, all of which are finite.

Then $\mathcal{G}(X)$ is finitely-generated.

<u>Proof:</u> Take a Postnikov system $\{X_n, p_n, \pi_n\}$ for which all k-invariants are zero. Then, we have $\mathcal{G}(X_{n-1}) = \mathcal{F}(X_{n-1})$ for all n. The theorem then follows from Lemma 3.1 and Theorem 2.1.

<u>Example</u>: Let $X = K(Z_2, 2) \times K(Z_2, 3)$. We then have $X_2 = K(Z_2, 2)$, $X_3 = X$. Clearly, $\mathcal{G}(X_2) = 1$. We have an exact sequence

$$1 \longrightarrow \ker(\rho) \longrightarrow \mathcal{G}(X_3) \longrightarrow 1 \longrightarrow 1$$

We determine $\ker(\rho)$. The homotopy class of f is determined by what it does to the two fundamental classes i_2 and i_3. We have

$$f^*(i_2) = i_2 \times 1$$
$$f^*(i_3) = 1 \times i_3 + \mathcal{P}(i_2) \times 1$$

where $\mathcal{P}(i_2)$ is zero or the non-zero element in $H^3(Z_2, 2, Z_2)$. Hence, $\ker(\rho) = \mathcal{G}(X_3) = Z_2$. Now, $H^3(X_3; Z_2) = Z_2 + Z_2$.

However, $\mathcal{G}_F(X_3)$ has only 2 elements. To see this, note that there are four maps (or homotopy classes)

$$h^* : X_3 \longrightarrow K(Z_2, 3)$$

according as

$$(h^*)^*(i_3) = 1 \times i_3,\ Sq^1\, i_2 \times 1,\ 1 \times i_3 + Sq^1\, i_2 \times 1, \text{ or } 0.$$

In each case, the homotopy equivalence would be

$$X \xrightarrow{h^* \times 1} K(Z_2, 3) \times X \xrightarrow{\bar{\mu}} X .$$

One readily calculates $(h^* \times 1)^* \bar{\mu}^*$ in the four cases. The respective answers are, on $(1 \times i_3)$:

$$0,\ Sq^1\, i_2 \times 1 + 1 \times i_3,\ Sq^1\, i_2 \times 1,\ 1 \times i_3$$

and on $(i_2 \times 1)$:

$$i_2 \times 1,\ i_2 \times 1,\ i_2 \times 1,\ i_2 \times 1$$

Hence, only in the second and fourth cases does one get a homotopy equivalence. Thus, $\mathcal{G}_F(X_3) = \ker(\rho) = Z_2$, and of course, $\mathcal{I}(X_n) = 1$. This last fact may also be checked directly.

Note that both elements in $\mathcal{G}(X)$ induce the identity on homotopy groups. We recall the following

<u>Def. 3.1</u> A space X is a Moore space of type (π, m); if

$$H_i(X; Z) = \begin{cases} 0, & i \neq m \\ \pi, & i = m \end{cases}$$

<u>Theorem 2.3</u> Let $\{X_n, p_n, \pi_n\}$ be any Postnikov system for X, where X is a Moore space of type (π, m), $m > 1$. We suppose $\mathrm{Ext}(\pi, \pi_{m+1}(X)) = 0$. Then for each n,

$$\mathcal{G}(X_n) \subset \text{Aut}(\pi)$$

as a subgroup.

<u>Proof:</u> By the Hurewicz Theorem, X is $(m-1)$ connected. $\mathcal{G}(X_m) = \text{Aut}(\pi)$. Suppose $\mathcal{G}(X_i) \subset \text{Aut}(\pi)$ for all integers i, such that $m \leq i \leq p$. By Theorem 2.1,

$$1 \longrightarrow \ker(\rho) \longrightarrow \mathcal{G}(X_{p+1}) \longrightarrow \mathcal{F}(X_p) \longrightarrow 1$$

is exact. $\mathcal{F}(X_p) \subset \mathcal{G}(X_p) \subset \text{Aut}(\pi)$. Hence it is sufficient to show that $H^{p+1}(X_{p+1};\pi_{p+1}(X)) = 0$. As X and X_{p+1} have the same $(p+1)$-type

$$H^{p+1}(X_{p+1};\pi_{p+1}(X)) \approx \text{Hom}(H_{p+1}(X;Z),\pi_{p+1}(X)) + \text{Ext}(H_p(X;Z);\pi_{p+1}(X)$$

$$\approx \text{Ext}\ (H_p(X;Z);\pi_{p+1}(X)) .$$

If $p = m$, this is zero by assumption. If $p > m$, $H_p(X;Z) = 0$.

As an application of this, we have

<u>Cor. 3.1</u> Let $\{X_n, p_n, \pi_n\}$ be a Postnikov decomposition of the sphere S^m, $m > 1$. Then

$$\mathcal{G}(X_n) = Z_2 , \quad n \geq m .$$

<u>Proof:</u> By Theorem 3.3, $\mathcal{G}(X_n) \subset \text{Aut}(Z) = Z_2$. Let $f : S^m \longrightarrow S^m$ be a map of degree -1. Then, $f_n : X_n \longrightarrow X_n$ induces a non-trivial automorphism on $\pi_m(X_n)$, so that we have $Z_2 \subset \mathcal{G}(X_n)$.

This corollary is of possible interest because relatively little information is available, at present, about the spaces X_n.

<u>Cor. 3.2</u> Let $\{X_n, p_n, \pi_n\}$ be a Postnikov decomposition of the

complex projective space $CP(m)$. Then $\mathcal{G}(X_n) = Z_2$, for $n \geq 2$.

<u>Proof:</u> $CP(m) = S^2 \cup e^4 \ \ldots \cup e^{2m}$

$$CP(\infty) = \bigcup_m CP(m) = K(Z,2) .$$

$\pi_2(CP(m)) = Z$, $\pi_i(CP(m)) = 0$ for $2 \leq i \leq m$. Hence, we have

$$H^n(X_n ; \pi_n(CP(m)) = 0, \quad i > 2 .$$

Thus, $\ker(\rho) = 1$, and $\mathcal{G}(X_n) \subset \mathcal{G}(X_{n-1}) \subset Z_2$. Of course, this corollary is just a special case of a similar result for the n-skeleton, $n > m$, of a space $K(\pi, m)$.

<u>Remarks:</u>

1. I would like to point out the following general problems about the groups $\mathcal{G}(X)$.

a.) What functorial properties does $\mathcal{G}(X)$ enjoy? As an example of the difficulties, let $\mathcal{C}$ denote the category of $K(\pi, m)$ spaces, $m > 1$ fixed, π an abelian group, and homotopy classes of maps. Homotopy classes $\{f\}$ are in 1-1 correspondence with group homomorphisms. The functorial properties of $\mathcal{G}$ on this category (if any) are the same as those of Aut on the category of Abelian groups.

b.) If $(E,F,B;p)$ is a fibre space, what are the relations between $\mathcal{G}(E)$, $\mathcal{G}(F)$ and $\mathcal{G}(B)$. Using the naturality properties of the Moore-Postnikov system, I can obtain information when F has finitely-many non-zero homotopy groups.

c.) What relations exist between $\mathcal{G}(X)$, $\mathcal{G}(SX)$ and $\mathcal{G}(\Omega X)$? What relations exist between $\mathcal{G}(X)$, $\mathcal{G}(A)$ and $\mathcal{G}(X/A)$, when $A \subset X$?

2. M. Atiyah (1) has determined $\mathcal{G}(B_U)$, as a consequence of his work on vector bundles. He shows that $\pi[B_U, B_U]$ is the image of the functor ρ of Borel and Serre (4). Along with Cor. 3.2, one ought to be able to determine $\mathcal{G}(X)$, when X is a complex Grassman manifold.

3. If X is a homotopy-Abelian H-space, $\pi[X,X]$ is an Abelian group. Using compostion, $\mathcal{G}(X)$ becomes a group of operators on $\pi[X,X]$, on the left or right. I hope to consider this case in a subsequent paper.

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FOOTNOTES

1. This work was partially supported by contract NONR 266(57).

2. Actually an isomorphism.

3. Remember that all spaces are assumed to have the homotopy-type of a 1-connected complex.

4. We interpret $F(X_{n-1})_I$ as the subgroup of $F(X_{n-1})$ whose elements have pre-images which induce the identity on homotopy.

5. It is essentially due to Minkowski.